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Critical behaviour of the discrete n-vector model in a planar self-dual lattice: a renormalization group approach

Edson de Pinho da Silva†‡, Ananias Monteiro Mariz§ and Constantino Tsallis†

† Centro Brasileiro de Pesquisas Físicas-CBPF/CNPq, Rua Dr Xavier Sigaud, 150, 22290 Rio de Janeiro, RJ, Brazil
‡ Departamento de Física, Universidade Federal Rural do Rio de Janeiro, Antiga Rio São-Paulo, KM 47, 23851 Rio de Janeiro, RJ, Brazil
§ Departamento de Física, Universidade Federal do Rio Grande do Norte, 59000 Natal, RN, Brazil

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Abstract. Within a real-space renormalization group which preserves two-site correlation functions, we study, for all values and signs of the coupling constants, the criticality of an extended version of the cubic model ($C(n)$). All results are exact for the associated planar self-dual hierarchical lattice The phase diagram typically exhibits five different phases. The n-evolution (for n real) of the thermal and crossover critical exponents is determined as well

In addition, we present an operational procedure (the break-collapse method) which considerably simplifies the exact calculation of the correlation function associated with two-terminal $C(n)$ graphs

1. Introduction

The cubic model has been used [1] to describe structural and magnetic phase transitions taking place in some rare-earth compounds. Aharony proposed [2] an extended version of it, henceforth called for simplicity the cubic model, which presents various interesting limiting cases (as will be seen in section 2). These models have been studied using several techniques (e.g. mean-field theories [1], dedecoration renormalization group (RG) [2], Niemeijer-van Leeuwen RG [3], Migdal-Kadanoff RG [4], variational RG [5], conformal invariance [6], Monte Carlo RG [7] and correlation-function-preserving RG [8]).

In all these approaches, only a restricted region of the parameter space is covered (positive parameters). In this paper we will establish, using a real-space RG technique, the phase diagram and some critical exponents of the model in *all* regions of the parameter space. We do this on a convenient hierarchical lattice.

In order to establish the RG equations, we develop for the cubic model an operationally easy method (referred to as the break-collapse method (BCM) which replaces the tracing operation by simple topological operations on some graphs. This type of procedure has been very useful in a variety of problems (the Potts model [9], resistor network [10], directed percolation [11], $Z(4)$ model [12] and $Z(n)$ model [13], among others). Herein we present the BCM and apply it to our problem.

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This paper is divided as follows: we present in section 2 the model and the formalism, in section 3 our results and finally we conclude in section 4.

2. Model and formalism

The cubic model is defined by the following dimensionless interaction Hamiltonian between spins i and j:

$$\beta\mathcal{H}_{ij} = -nK\mathbf{s}_i \cdot \mathbf{s}_j \tag{1}$$

where $\beta \equiv 1/k_B T$, $K \in \mathbb{R}$ and the spin $\mathbf{s}_i$ at any given site is an n-component unitary vector which can point only along the $2n$ positive or negative orthogonal coordinate directions, i.e., $\mathbf{s}_i = (\pm 1, 0, \ldots, 0)$ or $(0, \pm 1, 0, \ldots, 0)$ or $\ldots$ or $(0, \ldots, 0, \pm 1)$. This interaction is a discrete version of the classical n-vector model. The model we shall consider here is a generalized version of (1) that includes, besides the dipolar interaction, a quadrupolar one (see for instance [2]):

$$\beta\mathcal{H}_{ij} = -nK\mathbf{s}_i \cdot \mathbf{s}_j - n^2 L(\mathbf{s}_i \cdot \mathbf{s}_j)^2. \tag{2}$$

A direct motivation for studying (2) instead of (1) is that the quadrupolar term is generated under the RG transformation *even* if we start only with Hamiltonian (1), whereas Hamiltonian (2) is closed under the RG.

The model described by Hamiltonian (2), henceforth designated as the cubic model, contains as particular cases many interesting models: (i) in the limit $n \to 0$ [3], it reproduces the grand-canonical statistics of a self-avoiding walk (SAW) with step fugacity K; (ii) for $n = 1$ we have the spin-$\frac{1}{2}$ Ising model; (iii) the $n = 2$ case is the symmetric Ashkin–Teller model (or equivalently the $Z(4)$ model); (iv) if $nL = K$, we recover the $2n$-state Potts model with dimensionless coupling constant $2nK$; (v) if $K = 0$ we have the n-state Potts model with dimensionless coupling constant $n^2 L$; finally (vi) the limit $nL/|K| \to \infty$ (with finite K) recovers, for all values of n, the spin-$\frac{1}{2}$ Ising model with dimensionless coupling constant nK.

We study the cubic model on the hierarchical lattice generated in a standard way (see figure 1) where each bond of the cluster is substituted, in the next hierarchy, by the whole cluster. This particular cell is an excellent hierarchical approximation for the square lattice [14] because of its self-duality and because it correctly preserves, under the RG, the antiferromagnetic fundamental state.

Now in order to establish the RG recurrence equations in the (K, L) space we shall impose the preservation of the correlation function between the roots of both graphs shown in figure 1, i.e.

$$\exp(-\beta\mathcal{H}') = \operatorname*{Tr}_{\{\text{internal sites}\}} \exp(-\beta\mathcal{H}) \tag{3}$$

where $\mathcal{H}'$ and $\mathcal{H}$ are respectively the Hamiltonians associated with the small and the large graphs (in fact an additive constant must be included in $\mathcal{H}'$).

Although easy in principle, the tracing indicated in (3) is very tedious and we will not do it here. We shall instead use the BCM which we now describe.

The establishment of the BCM follows two steps: (i) definition of convenient variables (transmissivities) related to the original coupling constants present in the Hamiltonian; (ii) a set of algorithms to be applied for the particular graph we are interested in.

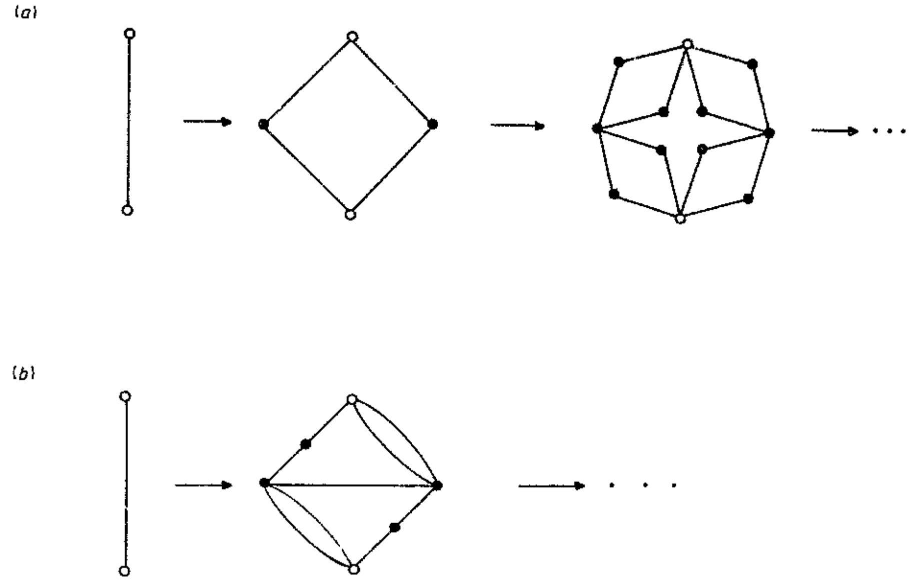

Figure 1. Hierarchical lattices: (*a*) illustration of the generating process for the diamond lattice, (*b*) lattice used in the present paper (● and ○ denote, respectively, internal and terminal sites of the graph)

Let us first introduce a convenient bond variable (the transmissivity [8]) for the cubic model, namely $t = (t_1, t_2)$ where

$$t_1 = \frac{1 - \exp(-2nK)}{1 + 2(n-1)\exp[-n(K+nL)] + \exp(-2nK)} \tag{4a}$$

$$t_2 = \frac{1 - 2\exp[-n(K+nL)] + \exp(-2nK)}{1 + 2(n-1)\exp[-n(K+nL)] + \exp(-2nK)}. \tag{4b}$$

Next we consider an arbitrary two-rooted (two terminal sites) graph whose equivalent transmissivity we denote by $t^{(G)} \equiv (t_1^{(G)}, t_2^{(G)})$ with $t_l^{(G)}(\{t^{(i)}\}) \equiv N_l(\{t^{(i)}\})/N_0(\{t^{(i)}\})$, $(l = 1, 2)$ where $\{t^{(i)}\}$ denotes the set of transmissivities respectively associated with bonds of the graph, and $N_l(\{t^{(i)}\})$ and $N_0(\{t^{(i)}\})$ are multilinear polynomials of the form $A + Bt_1^{(j)} + Ct_2^{(j)}$ for the arbitrarily chosen jth bond; A, B and C depend on the set of transmissivities (denoted $\{t^{(i)}\}'$) of the *remaining* bonds (i.e. the set $t^{(i)}\}$ excepting $t^{(j)}$). The performance of three different operations on the jth bond, namely, the 'break' $(t_1^{(j)} = t_2^{(j)} = 0$, which is equivalent to deleting that bond), the 'collapse' $(t_1^{(j)} = t_2^{(j)} = 1$, which is equivalent to contracting the two sites of the bond into a single site), the 'pre-collapse' $(t_1^{(j)} = 0;\ t_2^{(j)} = 1)$, and the use of the series and parallel algorithms (which we present explicitly further on) completely determines A, B and C. More specifically, we associate with each bond three numbers, namely

$$N_0^{(i)} = 1 \qquad N_1^{(i)} = t_1^{(i)} \qquad N_2^{(i)} = t_2^{(i)} \tag{5}$$

for the particular case of pre-collapsed bonds, this association is done as follows:

$$N_0^{(i)} = 1 \qquad N_1^{(i)} = 0 \qquad N_2^{(i)} = 1. \tag{6}$$

In order to evaluate the equivalent transmissivity of the graph we proceed as now described:

(i) Look for a series array of two or more bonds. For two bonds we use the algorithm

$$N_\alpha^{(s)} = N_\alpha^{(1)} N_\alpha^{(2)} \qquad (\alpha = 0, 1, 2). \tag{7}$$

(ii) Look for a parallel array of two or more bonds. For two bonds we use the algorithms

$$N_0^{(p)} = N_0^{(1)} N_0^{(2)} + n N_1^{(1)} N_1^{(2)} + (n-1) N_2^{(1)} N_2^{(2)} \tag{8a}$$

$$N_1^{(p)} = N_0^{(1)} N_1^{(2)} + N_0^{(2)} N_1^{(1)} + (n-1)[N_2^{(1)} N_1^{(2)} + N_1^{(1)} N_2^{(2)}] \tag{8b}$$

$$N_2^{(p)} = N_0^{(1)} N_2^{(2)} + N_0^{(2)} N_2^{(1)} + n N_1^{(1)} N_1^{(2)} + (n-2) N_2^{(1)} N_2^{(2)}. \tag{8c}$$

As an example consider two bonds with the respective transmissivities

$$t^{(1)} = \left(\frac{N_1^{(1)}}{N_0^{(1)}}, \frac{N_2^{(1)}}{N_0^{(1)}}\right) \qquad \text{and} \qquad t^{(2)} = \left(\frac{N_1^{(2)}}{N_0^{(2)}}, \frac{N_2^{(2)}}{N_0^{(2)}}\right).$$

If they are arranged in series, using (5) and (7) the equivalent transmissivity is given by

$$t^{(s)} = (t_1^{(1)} t_1^{(2)}, t_2^{(1)} t_2^{(2)}).$$

On the other hand if they are arranged in parallel, using (5) and (8) the equivalent transmissivity is given by

$$t^{(p)} = (t_1^{(p)}, t_2^{(p)})$$

where

$$t_1^{(p)} = \frac{t_1^{(1)} + t_1^{(2)} + (M-1) t_2^{(1)} t_1^{(2)} + (M-1) t_1^{(1)} t_2^{(2)}}{1 + M t_1^{(1)} t_1^{(2)} + (M-1) t_2^{(1)} t_2^{(2)}}$$

$$t_2^{(p)} = \frac{t_2^{(1)} t_2^{(2)} + M t_1^{(1)} t_1^{(2)} + (M-2) t_2^{(1)} t_2^{(2)}}{1 + M t_1^{(1)} t_1^{(2)} + (M-1) t_2^{(1)} t_2^{(2)}}.$$

Once we have solved all series and parallel arrays of bonds in our original graph, we choose any bond (denoted $t^{(j)}$) of the now simplified graph (the result does not depend on this choice) and perform on it the break, the collapse and the pre-collapse operations. Then, once more, we use (7) and (8) for the new series and parallel arrays of bonds. At this stage the transmissivity of the graph is given by the set of polynomials

$$N_\alpha = (1 - t_2^{(j)}) N_\alpha^{\mathrm{B}} + t_1^{(j)} N_\alpha^{\mathrm{C}} + (t_2^{(j)} - t_1^{(j)}) N_\alpha^{\mathrm{PC}} \qquad (\alpha = 0, 1, 2) \tag{9}$$

where $\{N_\alpha^{\mathrm{B}}\}$ refer to the transmissivity of the graph obtained through the break operation; similarly, $\{N_\alpha^{\mathrm{C}}\}$ refer to the transmissivity of the graph obtained through the collapse operation and $\{N_\alpha^{\mathrm{PC}}\}$ refer to the transmissivity of the graph resulting from the pre-collapse operation.

The above procedure is repeated until we are finally left with graphs containing only pre-collapsed bonds. To solve these we use

$$N_0 = n^\kappa \qquad N_1 = 0 \qquad N_2 = n^\kappa \tag{10}$$

where κ is the cyclomatic number of the graph, i.e. κ = (number of bonds) − (number of sites) + 1. By applying this procedure to figure 1 we obtain

$$t_1^{(G)} = N_1^{(G)} / N_0^{(G)} \qquad \text{and} \qquad t_2^{(G)} = N_2^{(G)} / N_0^{(G)} \tag{11}$$

where

$$N_1^{(G)} = 2x_1x_2 + 2(n-1)x_1x_2y_1y_2 + t_1(x_1^2+x_2^2) + (n-1)^2 t_1(x_1^2y_2^2 + x_2^2y_1^2)$$
$$+ (n-1)t_1(2x_1^2y_2 + 2x_2^2y_1) + (n-1)t_2(2x_1x_2y_1 + 2x_1x_2y_2)$$
$$+ (n-1)(n-2)t_2(2x_1x_2y_1y_2)$$
$$N_2^{(G)} = nx_1^2x_2^2 + 2y_1y_2 + 2nt_1x_1x_2(y_1+y_2) + n(n-1)t_2x_1^2x_2^2 + t_2(y_1^2+y_2^2) + (n-2)y_1^2y_2^2$$
$$+ n(n-2)t_1(2x_1x_2y_1y_2) + (n-2)(n-3)y_1^2y_2^2t_2 + (n-2)t_2(2y_1^2y_2 + 2y_2^2y_1).$$
$$N_0^{(G)} = 1 + nx_1^2x_2^2 + (n-1)y_1^2y_2^2 + 2nt_1x_1x_2 + 2n(n-1)t_1x_1x_2y_1y_2 + n(n-1)t_2x_1^2x_2^2$$
$$+ 2(n-1)t_2y_1y_2 + (n-1)(n-2)t_2y_1^2y_2^2$$

with

$$x_1 = t_1^2 \qquad y_1 = t_2^2$$
$$x_2 = \frac{2t_1 + 2(n-1)t_1t_2}{1 + nx_1 + (n-1)y_1} \qquad y_2 = \frac{2t_2 + nx_1 + (n-2)y_1}{1 + nx_1 + (n-1)y_1}.$$

By now identifying $t^{(G)}$ with t' (renormalized transmissivity), (11) provides the RG recursive relations we are looking for. Here we emphasize that we have obtained the desired RG equations using exclusively the algorithm stated in (7), (8) and (10) together with definitions (5) and (6). So, by doing simple topological operations on the elementary cell (cluster) of the hierarchical lattice, we avoid doing the very tedious traditional tracing operation indicated in (3). The proofs associated with the BCM have been presented by de Magalhães and Essam in [15]. For fixed n, the flow in the (K, L) space (or equivalently in the (t_1, t_2) space) will determine the phase diagram as well as the universality classes.

The critical exponents (thermal (ν) and crossover (ϕ)) are obtained by making (11) linear in the neighbourhood of the unstable fixed points. Denoting the eigenvalues of the corresponding jacobian matrix by λ_1 and λ_2 we have:

(i) $\lambda_1 > 1 > \lambda_2$ for critical (semi-stable) fixed points and

$$\nu = \ln b / \ln \lambda_1 \tag{12}$$

where b is the linear expansion factor ($b = 3$ for figure 1);

(ii) $\lambda_1 > 1$ and $\lambda_2 > 1$ for multicritical (fully unstable) fixed points,

$$\nu_s = \ln b / \ln \lambda_s \qquad (s = 1, 2) \tag{13}$$

and

$$\phi = \ln \lambda_2 / \ln \lambda_1 \tag{14}$$

where λ_2 denotes that eigenvalue which, as n varies, tends to unity whereas λ_1 remains greater than unity.

3. Results

The phase diagrams for typical values of n are shown in figures 2(a)–(d) (respectively corresponding to $n = 1, 2, 3$ and 15). The symmetries that we observe in figures 2 and 3 come from the fact that, for the graphs of figure 1 with Hamiltonian (2), the $K > 0$ and $K < 0$ cases are isomorphic. Since K and L are real quantities, the physical region

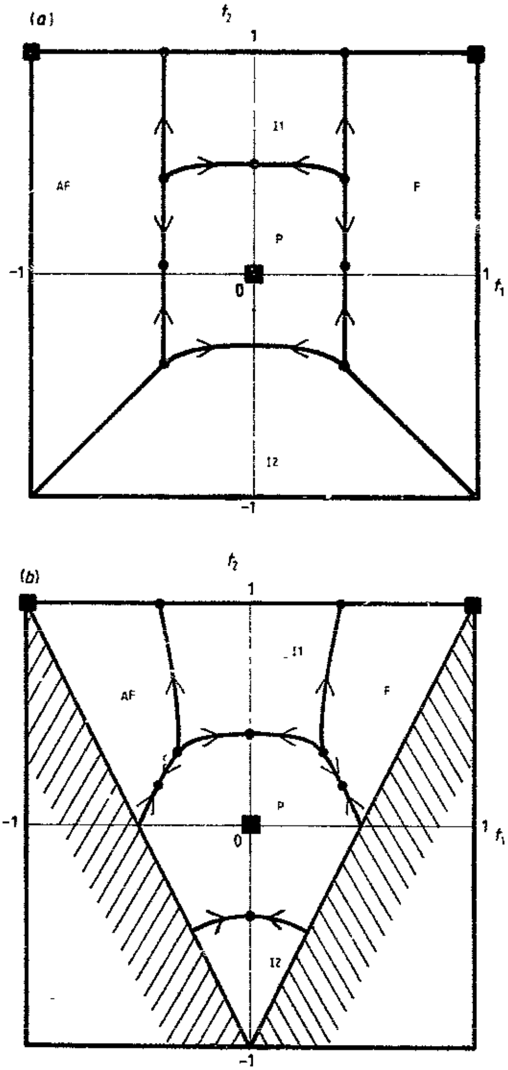

Figure 2. Phase diagram in (t_1, t_2) space for (a) $n=1$, (b) $n=2$, (c) $n=3$ and (d) $n=15$. The arrows indicate the RG flows; ■ and ● indicate, respectively, stable and unstable fixed points. The line $t_1=t_2$ corresponds to the $2n$-state Potts model The dashed region denotes unphysical parameters

in the (t_1, t_2) space satisfies simultaneously

$$1+nt_1+(n-1)t_2\geqslant 0 \qquad 1-nt_1+(n-1)t_2\geqslant 0 \qquad 1-t_2\geqslant 0. \qquad (15)$$

Our $n=2$ results precisely recover those of [16]; they are consistent with available Monte Carlo results [17]. The phase diagram exhibits the following five phases: (i) ferromagnetic phase, denoted F ($\langle S_i^z\rangle\neq 0$ and $\langle (S_i^z)^2\rangle\neq 0$); (ii) antiferromagnetic phase,

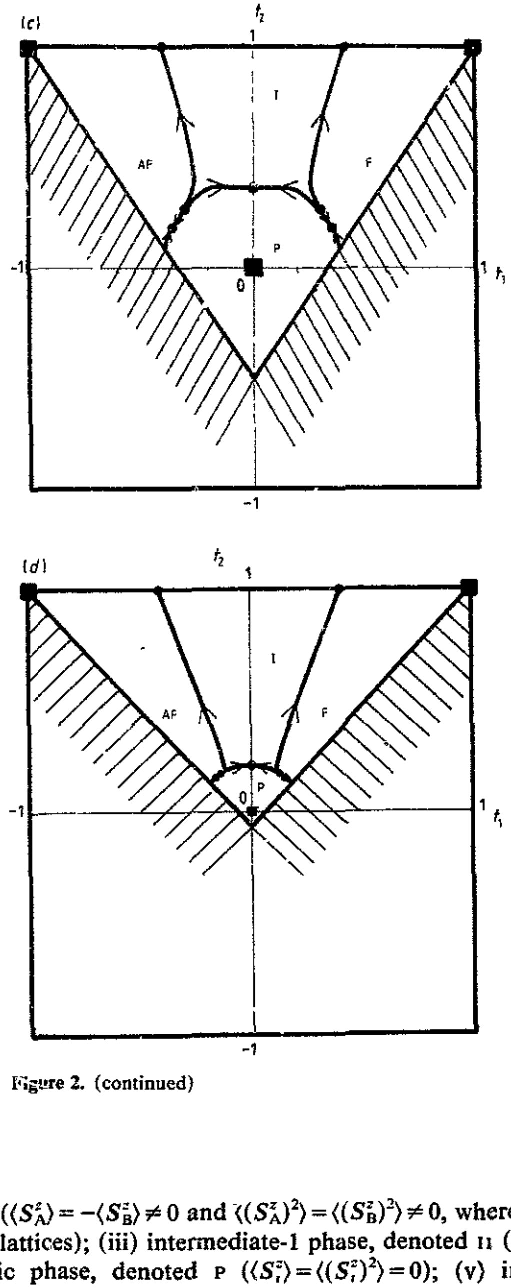

Figure 2. (continued)

denoted AF ($\langle S_A^z\rangle = -\langle S_B^z\rangle \neq 0$ and $\langle (S_A^z)^2\rangle = \langle (S_B^z)^2\rangle \neq 0$, where A and B are first-neighbouring sublattices); (iii) intermediate-1 phase, denoted I1 ($\langle S_i^z\rangle = 0$; $\langle (S_i^z)^2\rangle \neq 0$); (iv) paramagnetic phase, denoted P ($\langle S_i^z\rangle = \langle (S_i^z)^2\rangle = 0$); (v) intermediate-2 phase (or equivalently 'perpendicular order phase' for $n=2$ (POP phase)), denoted I2 ($\langle S_A^z\rangle = \langle S_B^z\rangle = 0$ and $\langle (S_A^z)^2\rangle = -\langle (S_B^z)^2\rangle \neq 0$).

For $n > n^* = 2.4$ the I2 phase disappears. The $n=1$ Ising model phase diagram exhibits an unphysical critical line, namely the P-I one, which must be considered as a mathematical artifact [8].

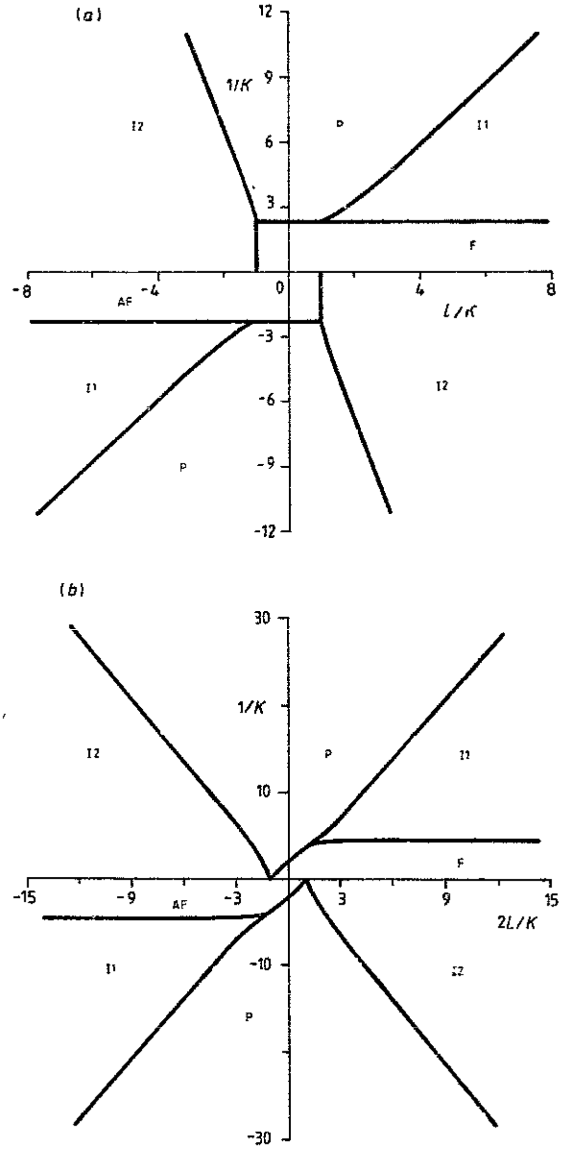

Figure 3. Phase diagram in the $(1/K, nL/K)$ space for (*a*) $n=1$, (*b*) $n=2$ and (*c*) $n=3$.

The generic situation for the critical lines for arbitrary n exhibits five physically non-equivalent fixed points, namely the n-state Potts ferromagnet $((t_1, t_2) = (0, 1/(\sqrt{n}+1)))$, the $2n$-state Potts ferromagnet $(t_1 = t_2 = 1/(\sqrt{2n}+1))$, the Ising ferromagnet $((t_1, t_2) = (\sqrt{2}-1, 1))$, the cubic ferromagnet $((t_1, t_2) = (t_1^c, t_2^c)$, where (t_1^c, t_2^c) vary continuously with n) and the n-state Potts antiferromagnet $((t_1, t_2) = (0, t_2^{AF})$, where t_2^{AF} varies continuously with n.

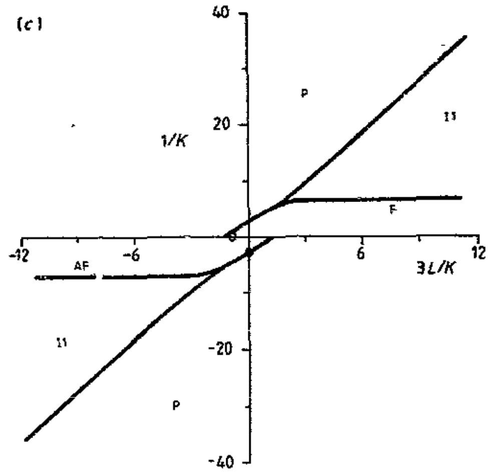

Figure 3. (continued)

Another fact to be mentioned is that at $n = n^{**} \simeq 8$, the $2n$-state Potts ferromagnetic fixed point and the cubic ferromagnetic fixed point collapse and interchange stability. The thermal and crossover critical exponents (ν and ϕ respectively) are shown in figure 4 for the $2n$-state Potts and extended cubic ferromagnet. In particular, in the limit $n \to 0$ (SAW), we have $\nu_T = \ln 3/\ln(69 - 16\sqrt{17}) \simeq 0.99$.

4. Conclusion

We have considered, within a real space RG framework, the criticality of the extended cubic model in a planar self-dual hierarchical lattice for all values of the coupling constants. The renormalization leaves the extended standard cubic model invariant; hence all present results are exact for the hierarchical lattice, and believed to be a good approximation for the square lattice (as long as first-order phase transitions are not concerned). For $n = 2$ our results are consistent with the Monte Carlo approach.

We have also verified two interesting properties of the model as the number of states varies, namely:

(i) The n-state Potts antiferromagnetic fixed point varies continuously with n, disappearing for values of $n > n^* \simeq 2.4$; in other words, the I2 phase disappears,

(ii) The $2n$-state Potts and cubic fixed points move towards each other, finally collapsing at $n = n^{**} \simeq 8$ and interchanging stability henceforth.

In addition, we have established a new method, simpler than the tracing operation, to calculate the equivalent transmissivity of an arbitrary two-terminal graph. Using this method based on elementary topological operations on graphs, RGs based on relatively large clusters become tractable.

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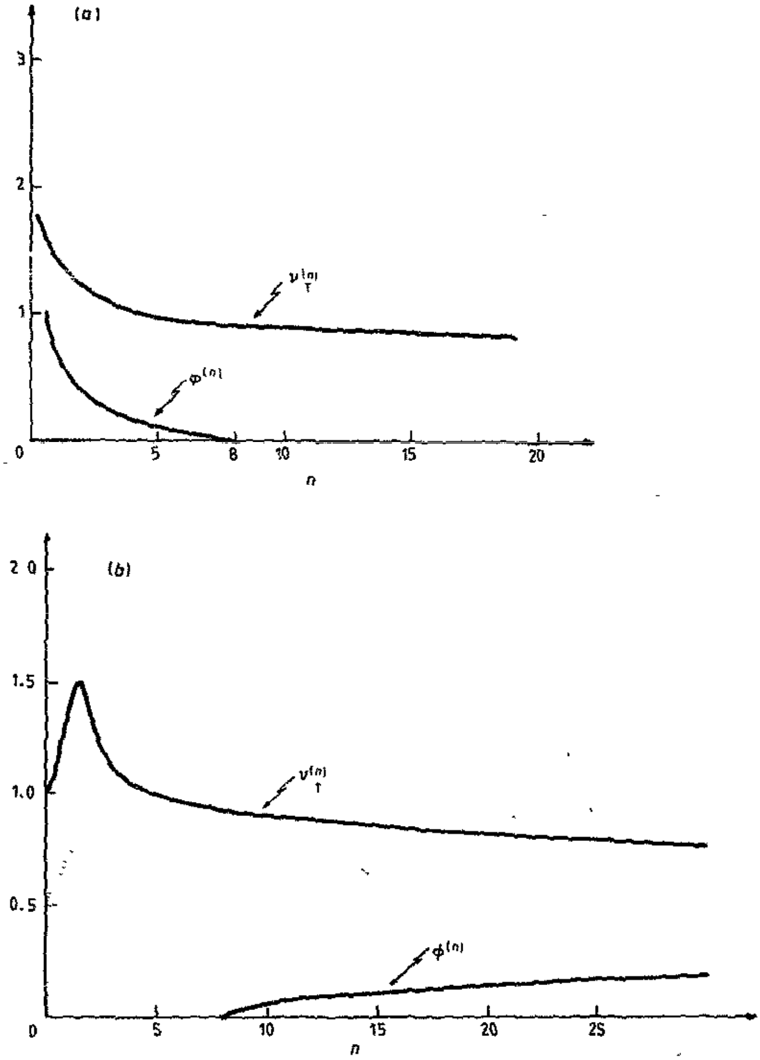

Figure 4. The n-dependence of the thermal critical exponent ν and the crossover exponent ϕ· (a) $2n$-state Potts model; (b) extended cubic model.

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